

DECISION MAKING AND NEW MATHEMATICS¹

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Anyone familiar with the history of science is aware that the development of physics and the development of mathematics are intimately connected. For example, Newton's Principia Mathematica contained, besides the physical theories with which his name is most often connected, also the origins of calculus and some very interesting investigations of algebraic curves (pure mathematics). Many other examples of this influence can be cited. It also is the case, although less well known, that problems of decision making have changed the course of mathematics. I am not prepared to give a historical study of this influence but rather to discuss some current problems of decision making, stimulated by the social or behavioral sciences, in particular economics, as they are influencing the creation of new mathematics today.

What I want to do here is to discuss a dozen or so new kinds of problems, arising directly or indirectly from decision making questions, which have been studied by one or more mathematicians. I will accept the assertion that if some new mathematical theorems have been discovered as the result of such studies, then these new problems are changing the course of mathematics.

Surprisingly enough, the problems to be discussed are of very recent origin. Indeed, the last fifteen years suffice to cover their publication dates, although their roots in many cases go back much further.

It is not possible for me to go into the technical details of the problems here. But I can illustrate them by means of simple examples and in this manner present an idea of the wide range of decision making problems that are being studied at the present time.

The first class of problems to be discussed are the so-called "games against Nature." A game against Nature is basically a statistical problem which I illustrate by means of the following oversimplified example: Suppose that you are a farmer and must choose in a certain year whether to plant wheat or corn (you can plant only one of these). Suppose also that hot, rainy weather is good for corn and bad for wheat, and that cool, dry weather has the reverse effect (and these are the only possible kinds of weather). To be specific, let's assign some numbers to illustrate these effects, as in the table below.

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		Weather (Nature)	
		Hot, Rainy	Cool, Dry
Farmer	Corn	10	5
	Wheat	3	8

The numbers represent (say) the yield in dollars per acre under the conditions specified. Observe that corn is, in general, a more profitable crop than wheat but that, in a bad year for corn, wheat is more profitable than corn. How shall you (the farmer) act, assuming that you have absolutely no information as to how Nature will choose the weather in the coming year (there is no long range weather forecast, etc.).

There are essentially two different kinds of solutions that have been proposed by various people for the problem: (a) Make an assumption as to what Nature will do and then maximize your own position relative to this assumption. (b) Try to choose your course of action (strategy) so that, regardless of what Nature does, you have in some sense protected yourself against the worst that could happen.

As representative of the decisions of type (a) we mention the Laplace decision rule: Choose whichever row has the largest average. A probability rationale for the Laplace rule can be given as follows: in the absence of any information as to what probability Nature has placed on the kinds of weather, assume that they are equally likely; e.g., in our example above assume that the probability of hot, rainy or cool, dry weather is each equal to one half; then compute the mathematical expectation* that results from each course of action under this assumption and act to maximize this expectation.

In the example above, the farmer's mathematical expectation for a given course of action, that is, for the choice of a given row, is the average of that row. The first row (plant corn) has the larger average, namely, 7.5; hence our decision, using the Laplace criterion, is always to plant corn.

The Wald and Savage decision rules are of type (b). Let us consider the Wald or minimax rule first. It is the following: Assume that the table of payoffs is that of a two-person zero-sum game and use as a decision rule an optimal strategy for that game. In our example above, the optimal strategy that the farmer should use is to choose each row with probability one half; for example, he could flip a coin and choose the first row if (say) heads turns up and choose the second if tails turns up. If the farmer uses this strategy year after year, then about half the time he will plant corn and half the time wheat. In this case, regardless of what weather Nature actually chooses, the farmer's expectation is 6.5, i.e., his average earnings over the years are 6.5 dollars per acre.

The Savage or minimax regret rule can be described as follows: Suppose that you knew what Nature's choice of a column would be; then define as your regret for having chosen a given course of action, that is a given row, the difference between the entry you receive in the indicated column and the maximum entry in that column. For the example above, the table of regrets is

*To explain the concept of mathematical expectation which we shall use frequently in this paper, suppose that there are n events of which one, and only one, can happen; suppose they give to us payoffs (monetary) in the amounts $a_1, a_2, \dots, a_n$; suppose that we know there are probabilities assigned to the occurrence of these events, say, $p_1, p_2, \dots, p_n$, where $p_i \geq 0$ and $p_1 + p_2 + \dots + p_n = 1$; then our mathematical expectation E is

$$E = a_1p_1 + a_2p_2 + \dots + a_np_n.$$

0	-3
-7	0

The minimax regret rule now is: Regard the regret table as a two-person zero-sum game and use as a decision rule an optimal strategy in that game. For our example the minimax regret rule gives the result: Plant corn with probability 0.7 and wheat with probability 0.3.

Observe that each of these decision rules gives a different answer. There are several other decision rules which give still other different answers. Each of these rules satisfies certain desirable properties, but none satisfies all desirable properties. For further discussion we refer the reader to Milnor's paper [9], on which the present discussion has been based.

The Wald decision rule which we have illustrated for a finite matrix game has much more general interpretations in the Theory of Statistical Decisions. There a pure strategy of the experimenter is to choose a given decision rule as a method of testing a hypothesis, and a pure strategy of Nature is the choice of a distribution. Relative to a choice of a decision rule by the experimenter and a distribution function by Nature there is a payoff, called the risk, that gives the cost of experimentation for that case. Then the decision problem is reduced to an infinite matrix game where Nature has at most a countable number of pure strategies. An optimal strategy in this game provides Wald's solution to the statistical decision problem.

Continuing our discussion, let us briefly consider the dynamic programming problems of R. Bellman [2]. Here an example will suffice to state the problem. Suppose that we are the owner of two gold mines of which one is a surface mine yielding very low grade ore and the other is a deep mine yielding high grade ore. The surface mine is easy and inexpensive to work but has low yield, while the deep mine is difficult and expensive to work but has high yield. Suppose that we have a fixed amount of capital which we desire to spend on the mining operations. At the beginning of each time period, which may be a month or a year, etc., we divide the available capital into two parts and use one part for surface mining and the other for deep mining. At the end of each time period a certain part of the capital remains for use during the next time period. The problem is to decide how to allot the portions of capital so that our returns are maximized.

The main difference between this and the preceding problems is that the present problem involves decisions made repeatedly over several time periods. Possibly even an infinite sequence of decisions is needed, each depending in some way on past results and an estimate of what will happen in the future. The most general such problem that could be stated is insoluble, but Bellman has shown the existence of optimal decision rules under certain simplifying assumptions.

The problems discussed so far belong to a class called one-person games. Thus in each of them there was only one decision-maker who controlled some, but not all, of the variables in the problem. The rest of the variables were controlled by Nature, about whose actions usually no a priori assertion could be made. Thus Nature could or could not be a malevolent participant in the game. The most conservative or pessimistic assumption about her behavior was that she was malevolent, and that assumption lead to minimax decision procedures. More optimistic assertions about Nature's behavior lead to other, less pessimistic, decision rules.

We now turn to the class of so-called two-person zero-sum games in which there are two active participants. The initiators of this, as well as of most other parts of the theory of games, are von Neumann and Morgenstern [10]. The minimax theorem of the theory of games assures that a solution, in the form of optimal (mixed) strategies for each player and an

(expected) value for the game, exist in all cases. Since there is some confusion as to the meaning of the minimax theorem, let me illustrate it by means of the customary example of matching pennies. The diagram below, where we have named the players Jones and Smith, gives the explanation.

		Smith	
		Head	Tail
Jones	Head	1	-1
	Tail	-1	1

Here the entries in the table are the payoffs which Jones receives from Smith (we assume that Jones is trying to match Smith). The value of the game is zero, and optimal strategies for either player are to choose head or tail with equal probability, i.e., with probability $1/2$. How can we interpret the value (zero) of the game? Observe that in a single play of the game each player will either win or lose one penny, whereas the value of zero might seem to indicate that neither player could either lose or win. Here we must remember that the word value means expected value in the sense of probability. Thus if the game is played many times one can say that with high probability the average earnings of each player become close to zero. In a single play of the game one can say that the optimal strategies make the probability of loss equal to the probability of win.

It is well known that players of this game realize their optimal strategy by "flipping" their coin, that is, leaving their choice of head or tails to a chance device that chooses head and tail with probability $1/2$. Thus the player himself does not know beforehand what choice he will make. Here is a case where spying would help. For example, if Jones could find out what choice Smith is going to make he could adjust his own choice accordingly and always win. Smith can prevent Jones from finding out (short of some psycho-kinetic effect) by leaving his choice up to the chance device and, in fact, not knowing himself exactly what choice he will make.

The idea that "playing a game optimally" should involve "leaving your choice up to a chance device" seems flippant to many people, particularly when game theory is applied to more important kinds of games than matching pennies. Another common criticism of the results of the minimax theorem is the following: Suppose that Smith is stupid enough always to choose the same side of his coin, say heads. In the course of playing the game several times Jones would certainly observe this behavior and, one would think, should take advantage of it by always himself choosing heads. But the latter behavior is not optimal! Indeed, if Jones plays optimally, then his expectation remains zero regardless of what Smith does, and in effect he is protecting Smith from the follies of his own (Smith's) stupidity!

The answer to the first criticism is that the minimax theorem proves that no other behavior can do better in all situations than the optimal behavior. It shows that the flipping of a coin, or the use of similar chance devices, is actually a serious and important way of making strategic decisions. The answer to the second criticism can be made in two ways. Some theorists would state that in order to take advantage of an opponent's mistakes it is necessary to deviate from an optimal strategy and take risks, and risks of that sort are not a proper subject of study in game theory. Another answer might be that if Smith plays in the stupid manner described above and Jones finds out about it, then Jones can suitably redefine the game for himself (e.g., cross out the second column) and play minimax in the redefined game. The optimal strategy in the new game is the one suggested above.

The theory of linear programming was developed by G. Dantzig and others about 1948. The classical example of a linear programming problem is the diet problem. Everyone must eat to live and must obtain minimal requirements of certain nutrients to remain healthy. Also, most people have limited incomes and hence wish to satisfy their diet requirements while at the same time minimizing their grocery bills. The amounts of nutrients contained in foods varies linearly with the quantity purchased and, if the quantities are small, so does the cost of the foods. Thus the problem is: What should our diet be in order that we obtain our minimum diet requirements and pay least for groceries? If the palatability of our food also is important we can add the requirement that at least certain minimum amounts of certain foods be eaten.

Many other examples of linear programming problems can be cited from applications to military, economic, and business problems.

The solution of a linear programming problem can be shown to be essentially equivalent to the solution of a two-person zero-sum game. The mathematical definition of a linear programming problem is that it requires the maximization or minimization of a linear function of n variables whose domain is a convex set of values. Because solutions of such problems occur on the boundaries of the convex set, it is not possible to use the ordinary techniques of calculus to find them. Instead, finite methods that can be programmed on high-speed electronic computers are used.

As another application of two-person zero-sum game theory to decision problems, I would like to mention some very recent work by John Kemeny, Oskar Morgenstern, and myself [7]. We considered a generalization of the von Neumann model of an expanding economy and found that it was intimately connected with the theory of games. It turned out that, although the building blocks of the model were ordinary economic concepts such as inputs, outputs, interest rates, expansion factors, etc., the type of mathematics needed to analyze the model and solve the problems connected with it was the theory of games. In other words, although game theory was not used at all in the formulation of the model, the mathematics that had been developed for game theory turned out to be precisely the kind of tool needed for its analysis. As far as I know this is the first time that the mathematics of game theory has been so applied. We found that in the generalized von Neumann model, which is a very general type of economic input-output model, there can be at most a finite number of expansion rates at which the economy can expand in equilibrium, and for each of these expansion rates, the interest rate is equal to the expansion rate.

We turn now from the theory of zero-sum two-person games to nonzero-sum two-person games. In so doing we leave a well-ordered subject that is dominated by an intuitively acceptable theorem, the minimax theorem, and approach a subject in which there is no widely acceptable or accepted theory. In fact, the subject is such that there appears to be little hope at present of finding a universally acceptable theory.

The only general theorem that has been proved in this area is the one due to J. Nash concerning the existence of equilibrium points. Briefly, it says that in any two-person game there always exists an equilibrium point, that is, a pair of strategies, one for each player, having the property that neither player can improve his position in the game by changing his own strategy while the other player keeps his strategy fixed. In other words, at an equilibrium point neither player can by his own efforts improve his position. Actually the idea of an equilibrium-point solution goes back quite far in the economic literature—at least as far back as Cournot. There are, however, objections to equilibrium points as solutions of games, as the next example will illustrate.

Suppose that there are two entrepreneurs, specifically let us say gasoline station operators, whose places of business are located on opposite sides of the same street. We may as well call them Jones and Smith, following our previous example. Suppose that all other gasoline stations are located at least several blocks away from Smith and Jones so that their only immediate competition is with each other. We shall assume that there are only two prices which each can charge for his gasoline, call them high and low. At the beginning of each day they must decide which price they will charge for the day and they cannot change price during the course of the day. The strategic decision each must make then is the price he will charge. Let the results of their decision be given in the diagram below:

		Smith	
		High	Low
Jones	High	(10,10)	(6,16)
	Low	(16,6)	(7,7)

In each pair of numbers enclosed in the parenthesis we indicate by the first entry the amount that Jones gets and by the second the amount that Smith gets. Let us see the rationale behind these numbers. If both charge the high price then each receives 10. If both charge the low price, which we can interpret as a "gas war," then, even though the volume of business may be greater, the low price cuts their income to 7 for the day. Finally, if one charges the high price and the other the low, the low-priced man can draw business not only away from his immediate competitor but also from elsewhere; and the volume of his business yields him 16 while the high-priced man gets only 6. Whether or not you agree that these numbers are realistic, I would like to keep them as chosen to illustrate several points about nonzero-sum games.

The equilibrium point in the game is (7,7), as can easily be checked. At this point each player charges the low price, i.e., the gas war prevails. If either player changes his strategy to that of charging the high price, he can only decrease his own income. To most people the solution seems irrational, since at the (10,10) point each player is better off than at the equilibrium point. But the (10,10) point is not stable, since either player can, by switching to the low price, improve his take from 10 to 16 (assuming that his opponent does not also change). It is also true that at the (6,16) or the (16,6) point the total amount they take in is greater than the total amount they take in at the (10,10) point, and so either of these points is preferable to the (10,10) point. Yet, again, neither of these two points is stable in the above sense.

The solutions that have been proposed for the game are the following: We have already mentioned the equilibrium point (7,7). The following cooperative solution has been proposed: Let one player, say Jones, always charge the high price and let Smith always charge the low price. Then Jones gets 6 and Smith gets 16 and their total take is 22. Now equalize the situation by having Smith make a side-payment of 5 to Jones and each gets a total take of 11. If there is some reason, moral or legal, why such a side-payment should not be made, then the players could adopt the following strategy: On alternate days let Jones charge high and Smith low and on the other days let the reverse be true. Then the average payment to each will be 11.

The scheme just described for mediating or arbitrating this two-person game is due to Nash and Raiffa and is also one of cooperative solutions proposed in the von Neumann-Morgenstern theory. It has been used in a very charming way by an English philosopher Braithwaite [3] to resolve another, somewhat similar situation, which he chooses to call a moral situation.

There is still another solution to this game which comes out of some work being done by M. Shubik and myself. Suppose that Jones is richer than Smith and suppose that at the equilibrium point (7,7) both players actually lose money. Since this game is played over and over again every day, we could imagine Jones making a threat to Smith as follows: "If you do not join with me in charging the high price, the day after you lower your price I shall lower my price and keep it down until you are bankrupt." Since Jones has more capital than Smith, the threat is effective and, if carried out, would result in the closing of Smith's business. Suppose, on the other hand, Smith tried to make a similar threat. Since he would go out of business first if he carried out the threat, it is not effective for him but merely suicidal. In this case Jones is able to enforce the (10,10) point and has the dominant position in what otherwise appears to be a symmetric game simply by being richer. We are continuing our work on the problem along this and other lines.

There are cases of three-and-more-person games which give illustrations as to the kinds of results that are being obtained there, but I will leave the theoretical realm and enter into the experimental realm by discussing some very unusual experimental results recently obtained by psychologists. These results will have a direct bearing on the concept of rationality, and I shall have a few comments about that concept later.

Let us digress a moment and talk about baseball. Suppose that, in a certain year, the following situation arose: The Yankees beat the Dodgers, the Dodgers beat the Giants, and the Giants beat the Yankees. Most probably the man in the street or the sports commentator would say that this was an "upset." It is also an example of an intransitive triple, namely, given three events, say A, B, and C, and an ordering relation "greater than" if it happens that A is greater than B, B is greater than C, and C is greater than A, then we say that A, B, and C form an intransitive triple. Similar intransitive relations also occur in the "pecking order" of chickens in a barnyard.

Most armchair thinkers about rationality assume that a "rational" person is consistent in the sense that his preference orderings for goods and services are transitive. Recent experimental work by Flood [6] and Papendreuou and theoretical work by Arrow [1] and May [8] have shown that when groups of people are concerned, there is a very real possibility of an intransitive ordering being set up by them. Even a single individual may have intransitive preferences, and I think that economists will soon have to include this possibility in their theories.

Another place where apparently irrational behavior on the part of individuals and groups of individuals has been observed is in the area of psychological learning theory. I shall describe an experiment from this area, one which I think has the most intriguing results of recent experimental psychology. Suppose that the subject (who may be a rat, a goldfish, a pigeon, or a human being) is placed in a room in which there are two "slot machines" and is instructed to play them. The experimenter has adjusted the machines so that one of them pays off 50 percent of the time and the other pays off never. What will the subject do? The experiment has been performed many times in many different guises and the results are essentially the same. Asymptotically it is observed that the subjects will tend to choose about $2/3$ of the time the machine that has the 50-percent payoff and about $1/3$ of the time the machine that never pays off. This result seems quite paradoxical since the game theory solution to the problem is always to choose the machine with the higher probability of payoff (after a few observations have established which is higher). Here is a case in which experiment gives different results than pure thinking. Learning theories which predict the observed result have

been devised by Bush and Mosteller [4] and Estes [5]. The mathematical study of these learning models has given rise to difficult mathematical problems, some of which are presently unsolved.

I can state another, to me astounding, result from experimental psychology, after a short digression. A man in the street, as well as the German philosopher K. Marbe, believes in the so-called gambler's fallacy, namely, that if in the repeated tossings of a true coin a long run (say 3 or 30 or 30,000,000) of heads have turned up, then the probability of a tail turning up the next time is larger than $1/2$. Experimental evidence does not confirm this belief. But if one takes a subject who indicates that he consciously believes in the gambler's fallacy and places him in an experimental situation of the kind described above, then, according to Estes, he will produce, in the course of his experimental run, a very nice random sequence—one containing no hint of the gambler's fallacy. We thus have the paradox of a subject who consciously disbelieves in random sequences, but who can produce, while apparently doing something else, very good random sequences.

I make the last two observations because I believe that in economics, and in other decision making fields, the day of armchair thinking about the rationality of the decision-maker is at an end. In attending meetings of professional economic societies I have observed that there is some reluctance on the part of economists to accept and include in their theories such empirical findings as the ones mentioned above. In my opinion, such results cannot be ignored and will have important influence on future economics.

In conclusion, let me say that in the course of this rapid survey of decision making I have mentioned several theories and indicated the kinds of mathematical problems to which they lead. In most cases, entirely new kinds of mathematics had to be developed and were developed by the cooperation of mathematicians and social scientists. There is no reason to think that this is a temporary trend or fad. On the contrary, I believe that it will continue to be of importance in the future.

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